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# Classifying Resolving Lists by Distances between Members

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## Abstract

Let  $G$  be a connected graph and let  $w_1, \dots, w_r$  be a list of vertices. We refer to the choice of a triple  $(r; G; w_1, \dots, w_r)$ , as a *metric selection*. Let  $\rho$  be the shortest path metric of  $G$ . We say that  $w_1, \dots, w_r$  *resolves*  $G$  (*metricly*) if the function  $c : V(G) \rightarrow \mathbb{Z}^r$  given by

$$x \mapsto (\rho(w_1, x), \dots, \rho(w_r, x))$$

is injective. We refer to  $c$  as the *code map*, and to its image as the *codes* of the triple  $(r; G; w_1, \dots, w_r)$ .

This paper proves basic results on the following questions:

1. What sets can be the image of a code map?
2. Given the image of a graph's code map, what can we determine about the graph?

Keywords: Metric Dimension; Distance in Graph

AMS Classification: Primary 05C12, Secondary 05C62

## 1 Introduction and Summary

We consider resolving lists, as used in the definition for metric dimension of finite graphs.

For  $k \in \mathbb{N}$ , let  $Ind(k)$  be the set of integers from 1 to  $k$ , inclusive.

Let  $G = (V(G), E(G))$  be a finite, simple, undirected, and (if necessary) connected graph of order  $|V(G)| \geq 2$ . Let  $\rho_G$ , or simply  $\rho$  if the context is clear, be the function such that  $\rho_G(x, y)$  is the length of a shortest path in  $G$  between vertices  $x$  and  $y$ .

Define a *metric selection* to be a triple  $(r; G; w_1, \dots, w_r)$  where  $r \in \mathbb{N}$ ,  $G$  is a connected graph, and  $w_1, \dots, w_r$  is an irredundant list of  $r$  vertices of  $G$ . Define the *code map* of this selection to be the function  $c$  from  $V(G)$  to  $\mathbb{Z}^r$  given by

$$v \mapsto (\rho(v, w_1), \dots, \rho(v, w_r)) .$$

We refer  $c(v)$  as the *code* of  $v$ . We say  $w_1, \dots, w_r$  is a *resolving list* of  $G$ , and that the metric selection is *resolved*, if its code map is injective.

The *metric dimension* of  $G$ , which is denoted here by  $\dim(G)$ , is the smallest  $r$  for which a resolving list of length  $r$  exists.

Our work arose from discussion of metric dimension, as characterized in **Chartrand et al.**[1] and **Harary et al.**[2]. The author contributed to **Eroh et al.**[4] which studies the impact of small changes on metric dimension. A coauthor of the latter paper, Dr. C.X. Kang, posed the question of how much of a graph is classified by its code map.

**Hernando et al.**[3] established an upper bound on the number of vertices of a graph of metric dimension  $r$  and diameter  $d$ . The authors demonstrated that the bound is sharp by building a graph based on a list of  $r$  vertices such that the distance between any two list members is  $b = \lfloor 2d/3 \rfloor + 1$ . Essentially, the text looks at all codes that could arise from such a graph, and then builds the graph using codes as vertices. The result is a graph whose order is the desired bound.

This paper does not deal with minimality questions. Instead, we attempt to classify graphs based on the image of the code map. Our classification begins by looking at

codes from a list of vertices for which the distances between the members (of the list) are fixed.

Suppose that  $c$  is a code map for a list  $w_1, \dots, w_r$  that resolves some graph  $G$ . For each  $i$ ,  $c(w_i)$  is the only member of the image of  $c$  that has 0 for its  $i$ -th component. Consequently, if one has just the image of  $c$ , one knows which codes represent the resolving list. Furthermore, the components in codes can now be identified as distances from a vertices whose codes are identified.

We start our classification with the basic information from codes of a resolving list.

**Definition 1.1.** Let  $r \in \mathbb{N}$ . By a *baseline matrix of dimension  $r$* , we mean a symmetric  $r \times r$  matrix  $B$  such that

- (1.a) for  $i, j \in \text{Ind}(r)$ ,  $B_{i,j}$  is a non-negative integer,
- (1.b) for  $i, j \in \text{Ind}(r)$ ,  $B_{i,j} = 0$  if and only if  $i = j$ , and
- (1.c) for  $i, j, k \in \text{Ind}(r)$ ,  $B_{i,j} + B_{j,k} \geq B_{k,i}$ .

Let  $B$  be such a matrix. Define

$$\text{Max}(B) = \max\{B_{i,j} : i, j \in \text{Ind}(r)\}.$$

Define a *metric type* to be a triple  $\tau = (r, B, h)$  where  $r \in \mathbb{N}$ ,  $B$  is a baseline matrix of dimension  $r$ , and  $h \in \mathbb{N}$  such that  $h \geq \text{Max}(B)$ .

Let  $r \in \mathbb{N}$  and let  $B$  be a baseline matrix of dimension  $r$ . Define  $U(B)$  to be the set of  $r$ -tuples  $(x_1, \dots, x_r)$  of non-negative integers such that

- (2.a) for all  $i, j \in \text{Ind}(r)$ ,  $x_i + x_j \geq B_{i,j}$ , and
- (2.b) for all  $i, j \in \text{Ind}(r)$ ,  $x_i + B_{i,j} \geq x_j$ .

The second condition is equivalent to requiring  $|x_i - x_j| \leq B_{i,j}$  for all  $i, j \in \text{Ind}(r)$ . For each  $i \in \text{Ind}(r)$ , let  $e_i(B) = (B_{i,1}, B_{i,2}, \dots, B_{i,r})$ . Trivially, each  $e_i(B) \in U(B)$ .

Let  $r$  and  $B$  be as before. Define  $\mu_B$  on  $U(B)^2$  by

$$\mu_B((x_1, \dots, x_r), (y_1, \dots, y_r)) = \max\{|x_i - y_i| : i \in \text{Ind}(r)\}.$$

Then  $\mu_B$  is a well-known metric. Define

$$E(B) = \{\{x, y\} : x, y \in U(B) \text{ such that } \mu_B(x, y) = 1\}.$$

Let  $\tau = (r, B, h)$  be a metric type. We define  $U(B, h)$ , or  $U(\tau)$ , to be the subset of all tuples  $(x_1, \dots, x_r) \in U(B)$  such that  $x_i \leq h$  for all  $i \in \text{Ind}(r)$ . For each  $i \in \text{Ind}(r)$ ,  $e_i(B) \in U(\tau)$ ; in this context, we refer to  $e_i(B)$  as  $e_i(\tau)$  or, if the meaning is clear from context,  $e_i$ . Likewise, we refer to the restriction of  $\mu_B$  to  $U(\tau)$  as  $\mu_\tau$ , or simply  $\mu$ .

Finally, define  $C(B, h)$  to be the graph whose vertex set is  $U(B, h)$  and whose set of edges consists of all pairs of  $U(B, h)$  members that belong to  $E(B)$ .

**Note:** Each member of  $U(B, h)$  serves two roles here. Literally, each is a tuple  $(x_1, \dots, x_r)$ . However, we want to build graphs in which each tuple is treated as vertex. We shall write “ $x = (x_1, \dots, x_r)$ ” as a sign that we treat the whole as an individual (usually a vertex)  $x$ , but need to cite the components  $x_i$  in our discussion.

Let  $\tau = (r, B, h)$  be a metric type, and let  $\alpha = (r; G; w_1, \dots, w_r)$  be a metric selection. We say that  $\alpha$  *has type*  $\tau$  if

(3.a)  $\forall i, j \in \text{Ind}(r)$ ,  $B_{i,j} = \rho_G(w_i, w_j)$ , and

(3.b)  $\forall i \in \text{Ind}(r)$ ,  $\forall v \in V(G)$ ,  $\rho_G(w_i, v) \leq h$ .

We leave it to the reader to verify that

(4.a) if  $(r; G; w_1, \dots, w_r)$  has type  $\tau$ , then the image of its code map is contained in  $U(\tau)$ , and

(4.b) every metric selection  $(r; G; w_1, \dots, w_r)$  is of some metric type  $\tau$ .

All of the inequalities required between members of  $B$  and coordinates of a tuple in  $U(B)$  follow from the Triangle Inequality for  $\rho$ .

Our main result is Proposition 2.1, which effectively characterizes which sets of tuples can appear as codes for a graph with a given baseline matrix. We postpone that statement until we have more language in place. A consequence is

**Corollary 1.1.** *Let  $\tau$  be a metric type.*

- (A) *Let  $(r; G; w_1, \dots, w_r)$  be a metric selection of type  $\tau$ . Let  $f$  be its code map. Then there is a metric selection  $(r; H; v_1, \dots, v_r)$  of type  $\tau$  whose code map  $p$  is injective and has the same image set as  $f$ .*
- (B) *Let  $(r; G_1; w_1, \dots, w_r)$  and  $(r; G'; w'_1, \dots, w'_r)$  be two metric selections of type  $\tau$ . Let  $f$  and  $f'$  be their respective code maps. Then there exists  $(r; H; v_1, \dots, v_r)$  of type  $\tau$  whose code map  $p$  is injective and whose image is the union of the images of  $f$  and  $f'$ .*

We conclude this introduction with facts about the  $C(B, h)$  graphs.

**Theorem 1.2.** *Let  $r, h \in \mathbb{N}$ , and let  $B$  be a baseline matrix of dimension  $r$  such that  $h \geq \text{Max}(B)$ . Then*

- (5.a)  $C(B, h)$  has diameter  $\leq h$ ,
- (5.b) the shortest distance metric of  $C(B, h)$  equals the restriction of  $\mu$ ,
- (5.c) the tuple  $(r; C(B, h); e_1, \dots, e_r)$  is resolved, and
- (5.d) the code map of  $(r; C(B, h); e_1, \dots, e_r)$  is the identity function.

Consequently, the triple  $(r; C(B, h); e_1, \dots, e_r)$  has a universal property among all metric selections of type  $(r, B, h)$ .

**Corollary 1.3.** *Let  $\tau = (r, B, h)$  be a metric type, and let  $(r; G; w_1, \dots, w_r)$  be a resolved metric selection of type  $\tau$ . Let  $f$  be the code map of  $(r; G; w_1, \dots, w_r)$ . Then the image of  $f$  lies in the set of vertices of  $C(B, h)$ . Furthermore, if  $xy \in E(G)$ , then  $f(x)f(y) \in E(C(B, h))$ .*

As noted earlier, the proof of [3][Lemma 3.3] can be roughly paraphrased as: for  $r, d \in \mathbb{N}$ , the graph of metric dimension  $r$  and diameter  $d$  that has the greatest number of vertices is based on a resolving list in which the distance between any two different members is roughly  $2d/3$ . This spurred the current author to ask: for a fixed baseline matrix  $B$ , is there a tight bound on the growth of  $|U(B, h)|$  as  $h \rightarrow \infty$ ? In fact, the growth is linear.

**Theorem 1.4.** Let  $r \in \mathbb{N}$  and let  $B$  be a baseline matrix of dimension  $r$ . Put  $h_0 = \text{Max}(B)$  and let  $h_1 = \lfloor (h_0 + 1)/2 \rfloor$ . Put  $h_2 = h_0 + h_1$ , and let  $n_1 = |U(B, h_2)|$ . Let  $S$  be the set of  $(x_1, \dots, x_r) \in U(B, h_2)$  such that at least one  $x_i$  equals  $h_2$ . Let  $n_2 = |S|$ . Then, for  $h \geq h_2$ ,

$$|U(B, h)| = n_1 + (h - h_2)n_2.$$

We conclude our discussion of  $C(B, h)$  by showing that, under minimal assumptions,

(6.a)  $C(B, h)$  has metric dimension  $r$  and

(6.b) every automorphism of this graph preserves the set  $\{e_1, \dots, e_r\}$ .

However, these results are limited in the following sense. Suppose that  $(r; G; w_1, \dots, w_r)$  and  $(r; H; z_1, \dots, z_r)$  are two resolved metric selections, both of which have type  $\tau = (r, B, h)$ . Furthermore, suppose that the images of their respective code maps happen to agree. It is possible that the diameters, the metric dimensions, and the automorphism groups of  $G$  and  $H$ , respectively, disagree.

Let  $W \subseteq V(C(B, h))$ , and let  $E \subseteq E(C(B, h))$  such that the endpoints of each  $e \in E$  belong to  $W$ . Definition 2.1 formulates the *subgraph coordinate condition* on the pair  $(W, E)$ . Proposition 2.1 shows that (1) each  $e_i \in W$  and (2) the code map of  $(r, (W, E); e_1, \dots, e_r)$  is the identity function.

It is easy to see that if  $W$  is fixed, there remains a huge variety of choices of  $E$  which satisfy this criterion. A few examples show that different choices produce wildly different graphs.

## 2 Proof of Theorem 1.2

Suppose  $\tau = (r, B, h)$  is a metric type. Let  $(r; G; w_1, \dots, w_k)$  be a metric selection of type  $\tau$ , and let  $f : G \rightarrow U(B, h)$  be its code map. If  $v, w$  are adjacent vertices of  $G$ , then for each  $i \in \text{Ind}(r)$ ,  $|\rho(w_i, v) - \rho(w_i, w)| \leq 1$ . Consequently,  $\mu(f(v), f(w)) \leq 1$ . This observation is the basis of our definition of  $E(B)$ .

Next, consider a subset of  $U(B, h)$  which is assigned a set of edges to make it a graph. The next theorem characterizes when such a graph creates a metric selection whose code map is the identity.

**Definition 2.1.** Let  $\tau = (r, B, h)$  be a metric type. Let  $H$  be a graph whose vertex set is a subset of  $U(B, h)$  and whose edge set is a subset of  $E(B)$ . We say that  $H$  satisfies the *subgraph coordinate condition* for  $\tau$  if the following is true:

- (7) Suppose that  $x = (x_1, \dots, x_n) \in V(H)$  and  $i \in \text{Ind}(r)$  such that  $x_i \neq 0$ . Then there exists  $y = (y_1, \dots, y_n) \in V(H)$  such that  $xy \in E(H)$  and  $y_i = x_i - 1$ .

Suppose  $W$  is a subset of  $U(\tau)$ . Let  $W^*$  be the graph on  $W$  induced from  $C(B, h)$ . If  $W^*$  satisfies (7), we say that  $W$  satisfies the *subset coordinate condition* for  $\tau$ .

We show that the subgraph coordinate condition on a graph  $H$  implies that  $H$  is connected. That is, connectedness is *not* assumed.

**Lemma 2.1.** Let  $\tau$  be a metric type. If  $x = (x_1, \dots, x_r) \in U(\tau)$  and  $i \in \text{Ind}(r)$  such that  $x_i = 0$ , then  $x = e_i$ .

*Proof.* Expand  $\tau = (r, B, h)$ . Let  $x = (x_1, \dots, x_n) \in U(\tau)$  and  $i \in \text{Ind}(r)$  such that  $x_i = 0$ . For each  $j$ , restrictions (2.b,c) become  $x_j \geq B_{i,j}$  and  $B_{i,j} \geq x_j$ , respectively.  $\square$

We can now prove that the subgraph coordinate condition implies that each tuple, as a vertex, is also its image under the code map. The full statement is

**Proposition 2.1.** *Let  $\tau$  be a metric type, and suppose that  $H$  satisfies the subgraph coordinate condition for  $\tau$ . Then*

(8.a)  $H$  is a connected graph,

(8.b)  $e_1, \dots, e_r \in V(H)$ ,

(8.c) the code map of  $(r, H; e_1, \dots, e_r)$  is the identity function, and

(8.d)  $\forall x, y \in V(H), \rho_H(x, y) \geq \mu(x, y)$ .

*Proof.* Put  $W = V(H)$  and  $\rho = \rho_H$ . For each  $i \in \text{Ind}(r)$ , there is  $x \in V(H)$  whose  $i$ -th coordinate is least amongst all members of  $H$ . The subgraph coordinate condition, plus Lemma 2.1, implies  $x = e_i$ .

For the next assertion, fix  $i \in \text{Ind}(r)$  and let  $D$  be the connected component of  $H$  that contains  $e_i$ . We induct on a double-implication: For each  $k \in \mathbb{N} \cup \{0\}$ , our proposition is that for  $x = (x_1, \dots, x_r) \in W$ ,

(9.a) if  $x_i \leq k$ , then  $x \in D$  and  $\rho(e_i, x) = x_i$ , and

(9.b) if  $x \in D$  and  $\rho(e_i, x) \leq k$ , then  $\rho(e_i, x) = x_i$ .

Lemma 2.1 proves the base case  $k = 0$ .

Assume  $k \in \mathbb{N} \cup \{0\}$  such that (9.a,b) both hold. Let  $y = (y_1, \dots, y_r) \in W$ .

We test the implications one at a time for  $k + 1$ . First, suppose  $y_i = k + 1$ . By the subgraph coordinate condition, there is a neighbor

$$x = (x_1, \dots, x_r) \in W$$

such that  $x_i = y_i - 1 = k$ . By the inductive hypothesis,  $x \in D$  and  $\rho(e_i, x) = k$ . Consequently,

$$y \in D \text{ and } \rho(e_i, y) \in \{k - 1, k, k + 1\}.$$

If  $\rho(e_i, y) \leq k$ , the second part of the inductive hypothesis would imply  $y_i \leq k$ , which is false. Therefore,  $\rho(e_i, y) = k + 1$ .

As before, assume  $y = (y_1, \dots, y_r) \in W$ . This time, we also assume that  $y \in D$  and  $\rho(e_i, y) \leq k + 1$ . Our goal is to show that  $\rho(e_i, y) = y_i$ . If  $\rho(e_i, y) \leq k$ , the inductive hypothesis applies to  $y$ , and we are done. Suppose  $\rho(e_i, y) = k + 1$ . Then  $y$  must have a neighbor  $x \in D$  such that  $\rho(e_i, x) = k$ . Expand  $x = (x_1, \dots, x_r)$ . Apply the inductive hypothesis to  $x$  to deduce that  $x_i = k$ . Now  $xy \in E(H)$ , which means  $\mu(x, y) \leq 1$ . Therefore  $y_i \in \{k - 1, k, k + 1\}$ . If  $y_i \leq k$ , the other half of the inductive hypothesis implies that  $\rho(e_i, y) = y_i \neq k + 1$ . Therefore,  $y_i = k + 1 = \rho(e_i, y)$ .

Assertions (a), (b) and (c) of Proposition 2.1 follow immediately.

Let  $x = (x_1, \dots, x_r)$  and  $y = (y_1, \dots, y_r) \in W$ . The triangle inequality implies that

$$\forall i \in \text{Ind}(r), \rho(x, y) \geq |\rho(e_i, x) - \rho(e_i, y)| = |x_i - y_i|.$$

Consequently,  $\rho(x, y) \geq \mu(x, y)$ .  $\square$

We add an informal corollary. Suppose  $G$  is a graph and  $w_1, \dots, w_r$  resolves it. Let  $f$  be the corresponding code map. Suppose we augment  $G$  by adding edges. That is, let  $E_1$  be a set of ordered pairs of vertices of  $G$ , and create a new graph  $G_1$  by adding  $E_1$  to the existing edges of  $G$ . Consider the question: when is  $f$  also the code map of  $G_1$ ? From Proposition 2.1,  $f$  remains the code map for  $G_1$  if and only if for every new edge  $ab \in E_1$ ,  $\mu(f(a), f(b)) \leq 1$ .

**Examples.** The above proposition characterizes which graphs can produce a specified set of codes. It also reveals that many aspects of the graph are not determined by the codes. We consider two simple examples. Let  $n$  be an integer  $\geq 3$ .

- Model the path graph  $Z_{2n}$  as consisting of integers  $(1 - n) \leq i \leq n$ , where successive integers are connected by an edge. Choose  $e_1 = 0$  and  $e_2 = 1$ . For  $1 \leq i \leq n$ , the code for  $i$  is  $(i, i - 1)$  and the code for  $1 - i$  is  $(i - 1, i)$ . By Proposition 2.1, the code map will be unchanged if we add one or more edges between  $i$  and  $1 - i$ . Now  $Z_{2n}$  has a unique non-trivial automorphism  $\sigma$  (in this case,  $i \mapsto 1 - i$ ) which preserves  $\{e_1, e_2\}$ . Adding one or more of the allowed edges produces a graph with the same automorphism; however, any one extra edge reduces the diameter from  $n - 1$  to a small value. If one adds just the edge from  $1 - n$  to  $n$ , graph mutates into  $C_{2n}$ . This new graph now has a transitive automorphism group.
- This time, consider the cycle  $C_{2n+1}$ , and choices  $e_1$  and  $e_2$  which are distance  $n$  apart. The codes from this choice consist of all pairs  $(i, j)$  in which  $i + j$  is either  $n$  or  $n + 1$ . Points with codes  $(i, n - i)$  form one arc from  $e_1$  to  $e_2$ , and the remaining codes form the other. Without altering the image of the code map, edges can be attached between  $(i, n - i)$  and  $(i + 1, n - i)$  and/or  $(i, n + 1 - i)$ . From the perspective of the other arc,  $(j, n + 1 - j)$  can be linked to  $(j, n - j)$  and/or  $(j - 1, n + 1 - j)$ . Therefore, the same code map applies to a variety of graphs based on  $C_{2n+1}$ , with a potentially large number of links between the two arcs. None of the new edges change the diameter, which remains  $n$ ; however, most additions create a graph whose automorphism group is trivial.

We can now prove assertions made in the introduction.

**Proof of Corollary 1.1** Fix a type  $\tau = (r, B, h)$ . For Part (A), let  $(r; G, w_1, \dots, w_r)$  be a metric selection of type  $\tau$ , and let  $f$  be its code map. Let  $W$  be the image of  $f$ , and let  $E$  be the set of edges of the form  $f(x)f(y)$  where  $xy \in E(G)$  (and  $f(x) \neq f(y)$ ). Clearly  $H = (W, E)$  satisfies the subgraph coordinate condition. By Proposition 2.1, the code map for  $(r; H; e_1, \dots, e_r)$  is the identity. In other words, the latter is resolved and has the same image as the code map for  $G$ .

Now suppose that  $(W_1, E_1), (W_2, E_2)$  both satisfy the subgraph coordinate condition for  $\tau$ . Obviously,  $(W_1 \cup W_2, E_1 \cup E_2)$  satisfies the same condition. An obvious variation on the previous paragraph proves Part (B).  $\square$

The following lemma implies most of the remaining verifications. It extends [3][Lemma 3.4].

**Lemma 2.2.** Let  $\tau = (r, B, h)$  be a metric type. Let  $x = (x_1, \dots, x_r)$  and  $y = (y_1, \dots, y_r)$  be different members of  $U(\tau)$ . Define an  $r$ -tuple  $z = (z_1, \dots, z_r)$  by

$$z_i = \begin{cases} x_i & \text{if } x_i = y_i, \\ x_i + 1 & \text{if } x_i < y_i, \\ x_i - 1 & \text{if } x_i > y_i. \end{cases}$$

Then  $z \in U(\tau)$ .

*Proof.* We run through the inequalities required of the coordinates of  $z$ . Obviously,  $z_i \leq h$  for each index  $i$ .

Now suppose  $i, j \in \text{Ind}(r)$  so  $i \neq j$ . We claim that  $z_i + z_j \geq B_{i,j}$ . First, if  $z_i = x_i + 1$  or  $z_j = x_j + 1$ , then

$$z_i + z_j \geq x_i + x_j \geq B_{i,j}.$$

Now suppose  $z_i \leq x_i$  and  $z_j \leq x_j$ . Then  $z_i \geq y_i$ ,  $z_j \geq y_j$ , and

$$z_i + z_j \geq y_i + y_j \geq B_{i,j}.$$

This completes one set of conditions.

Again, suppose  $i, j \in \text{Ind}(r)$  so  $i \neq j$ . This time, we argue that  $z_i + B_{i,j} \geq z_j$ . When  $z_i \geq z_j$ , this assertion is vacuous. Now assume  $z_j > z_i$ . First, if  $z_j = x_j - 1$ ,

$$z_i + B_{i,j} \geq x_i - 1 + B_{i,j} \geq x_j - 1 = z_j.$$

From now on, assume  $z_j \geq x_j$ .

If  $z_i = x_i + 1$ ,

$$z_i + B_{i,j} = x_i + 1 + B_{i,j} \geq x_j + 1 \geq z_j.$$

Now assume  $z_i \leq x_i$ .

In the remaining case,  $y_j \geq z_j$  and  $y_i \leq z_i$ . Then

$$z_i + B_{i,j} \geq y_i + B_{i,j} \geq y_j \geq z_j.$$

□

**Proof of Theorem 1.2** Let  $x \in U(B, h)$  and  $i \in \text{Ind}(r)$  such that  $x_i \neq 0$ . Let  $z$  be as in Lemma 2.2 for the ordered pair  $x, e_i$ . Then  $z_i = x_i - 1$ . Hence, the set  $U(B, h)$  satisfies the subgraph coordinate condition.

Let  $x, y$  be two different members of  $U(B, h)$ , and let  $z$  be the construction of Lemma 2.2. Then

$$(10.a) \quad \mu(x, z) = 1,$$

$$(10.b) \quad x \text{ and } z \text{ are adjacent in } C(B, h), \text{ and}$$

$$(10.c) \quad \mu(y, z) = \mu(y, x) - 1.$$

Based on this step, it is easy to set up an inductive proof that  $\mu(x, y) = \rho(x, y)$ . We leave the details to the reader. □



### 3 Proof of Theorem 1.4

Let  $r \in \mathbb{N}$  and let  $B$  be a baseline matrix of dimension  $r$ . Put  $h_0 = \text{Max}(B)$  and let  $h_1 = \lfloor (h_0 + 1)/2 \rfloor$ . Put  $h_2 = h_0 + h_1$ . For  $h \geq h_2$ , put

$$S_h = \{(x_1, \dots, x_r) \in U(B, h) : \exists i \in \text{Ind}(r), x_i = h\}.$$

Informally,  $S_h$  is the “outer shell” of vertices of  $U(B, h)$ . It consists of vertices that are distance  $h$  from at least one  $e_i$ . Furthermore,  $U(B, h+1)$  is the disjoint union of  $U(B, h)$  and  $S_{h+1}$ . Each  $+1$  increase to  $h$  corresponds to adding the shell  $S_{h+1}$  to  $U(B, h)$ .

We claim that  $|S_h| = |S_{h_2}|$  for every  $h \geq h_2$ . The equation of Theorem 1.4 follows directly. The claim can be established by induction. It suffices to show that  $|S_{h+1}| = |S_h|$  when  $h \geq h_2$ .

Define  $\theta : \mathbb{Z}^r \rightarrow \mathbb{Z}^r$  by

$$(x_1, \dots, x_r) \mapsto (x_1 - 1, \dots, x_r - 1).$$

Obviously,  $\theta$  is injective.

Let  $x = (x_1, \dots, x_r) \in \mathbb{Z}^r$ , and expand  $y = \theta(x) = (y_1, \dots, y_r)$ . Let  $h \geq h_2$ . For  $i$  any index,  $x_i = h + 1$  is equivalent to  $y_i = h$ . Also, for any  $i, j \in \text{Ind}(r)$ ,

$$|x_i - x_j| = |y_i - y_j| \quad \text{and} \quad y_i + y_j = x_i + x_j - 2.$$

It follows easily that if  $y \in S_h$ , then  $x \in S_{h+1}$ .

Conversely, suppose  $x \in S_{h+1}$ . In particular,  $x$  cannot be any  $e_i$ . Therefore, all entries of  $x$  are positive, and all entries of  $y$  are  $\geq 0$ . It is clear that  $y \in S_h$  provided that  $x_i + x_j - 2 \geq B_{i,j}$  for all  $i \neq j$ . The assertion reduces to proving that  $x \in S_{h+1}$  implies the latter condition.

Now fix  $x = (x_1, \dots, x_r) \in S_{h+1}$ . There is  $k \in \text{Ind}(r)$  such that  $x_k = h + 1$ . Obviously  $x_k \geq h_1 + 1$ . For  $i \in \text{Ind}(r)$  so  $i \neq k$ ,

$$x_i + B_{i,k} \geq x_k \Rightarrow x_i + h_0 \geq h_0 + h_1 + 1 \Rightarrow x_i \geq h_1 + 1.$$

Therefore, for  $i, j \in \text{Ind}(r)$  such that  $i \neq j$ ,

$$x_i + x_j - 2 \geq (h_1 + 1) + (h_1 + 1) - 2 \geq 2h_1 \geq h_0 \geq B_{i,j}.$$

We are done.  $\square$

### 4 Properties of Canonical Graphs

For this section, fix a metric type  $\tau = (r, B, h)$ . The study of metric dimension began with a search for resolving sets of minimal length. We can make some comments about minimality for our canonical graphs.

**Proposition 4.1.** *Let  $\tau = (r, B, h)$  be a metric type. Assume  $h \geq 4$  and  $B_{i,j} \neq 1$  for any  $i, j \in \text{Ind}(r)$ . Then  $C(B, h)$  has metric dimension exactly  $r$ .*

*Proof.* It is an easy exercise to see that  $h \geq 4$  implies existence of  $b \in \mathbb{N}$  for which

$$\frac{\sup(B)}{2} + 1 \leq b \leq h - 1 .$$

Let  $S \subseteq \mathbb{N}^r$  be the subset of  $(x_1, \dots, x_r)$  such that  $b-1 \leq x_i \leq b+1$  for each  $i \in \text{Ind}(r)$ . If  $B_{i,j} \neq 1$  for all  $i, j \in \text{Ind}(r)$ , then

(11.a)  $S \subseteq U(B, h)$ , and

(11.b) for  $\beta = (b, \dots, b)$ ,  $\beta \in S$  and  $S$  contains  $3^r - 1$  neighbors of  $\beta$ .

In a graph with metric dimension  $t$ , no vertex may have more than  $3^t - 1$  neighbors.  $\square$

Another counting trick proves

**Proposition 4.2.** *Let  $\tau = (r, B, h)$  be a metric type. Assume that  $h \geq 2$  and either  $r > 2$ , or  $r = 2$  and  $B_{1,2} \neq 1$ . Then any automorphism of  $C(B, h)$  preserves the subset  $\{e_1, \dots, e_r\}$ .*

*Proof.* Suppose that  $x = (x_1, \dots, x_r)$  and  $y = (y_1, \dots, y_r)$  are vertices of  $C(B, h)$  and  $\rho(x, y) = h$ . Then there is an index  $i \in \text{Ind}(r)$  such that

$$\{x_i, y_i\} = \{0, h\} .$$

Consequently, either  $x$  or  $y$  is  $e_i$ .

Now suppose  $x \notin \{e_i, \dots, e_r\}$ . Then the set of  $y$  for which  $\rho(x, y) = h$  is a subset of  $\{e_i, \dots, e_r\}$ . Specifically, there are  $\leq r$  possible choices for  $y$ .

Fix an index  $i \in \text{Ind}(r)$ , and let  $T_i \subseteq \mathbb{N}^r$  be the set of tuples in which

(12.a) the  $i$ -th coordinate is  $h$ , and

(12.b) every other coordinate is either  $h$  or  $h - 1$ .

As long as  $h \geq 2$ ,  $T_i \subseteq U(B, h)$ . There are  $2^{r-1}$  members of  $T_i$ , and every one is distance  $h$  from  $e_i$ . If  $r > 2$ , then  $2^{r-1} > r$ . If  $r = 2$  and  $B_{1,2} \neq 1$ , we make two observations:

(13.a) The set of  $y \in U(B, h)$  for which  $\rho(y, e_1) = h$  will include  $(h, h)$ ,  $(h, h - 1)$  and  $(h, h - 2)$ .

(13.b) A similar comment applies to vertices  $h$  units from  $e_2$ .

In all these cases, consider the subset

(14)  $D$  is the set of vertices  $x$  with the following property: the number of  $y$  at distance exactly  $h$  from  $x$  is  $> r$ .

If  $r > 2$ , or if  $r = 2$  and  $B_{1,2} \neq 1$ , the set  $D$  is exactly  $\{e_1, \dots, e_r\}$ .

Finally, suppose  $\alpha$  is an automorphism of  $C(B, h)$ . If  $x \in D$ , as defined above, it is easy to see that  $\alpha(x)$  is in the same subset  $D$ . But under the hypothesis,  $D$  is exactly  $\{e_1, \dots, e_r\}$ .  $\square$

The above results share a limitation. Both rely on the using every allowable edge  $C(B, h)$ . Proposition 2.1 shows that there may be a graph  $G$  whose vertex set is  $U(B, h)$ , whose code map is the identity, but which has far fewer edges than the canonical choice. (It is easy to see that there will usually be many such  $G$ .) There

is no assurance that  $\beta$  of (11.a) must have so many neighbors in  $G$ . So the metric dimension could drop.

Although the code map is unchanged, the metric  $\rho$  for a reduced  $G$  might not agree with metric for  $U(B, h)$ . Our proof of Proposition 4.2 is based on a property phrased using distance.

## 5 Back to Classification

The challenge that began this paper was to classify a graph in terms of its codes. Our theorems here do not create a classification; instead, they set up a language in which to discuss issues of classification. We illustrate how any resolved graph is equivalent to a graph on which the code map is the identity function. We have also characterized when a set of edges between codes creates such a graph. As a consequence, we show that it is easy to make new graphs from codes of two initial graphs.

What sort of properties can be deduced from a set of codes? We conclude this paper with a sample observation.

Let  $G$  be a connected graph, and let  $v \in V(G)$ . Define  $Ev(G, v)$  and  $Odd(G, v)$  to be, respectively, the set of vertices whose distance from  $v$  is even and whose distance is odd. Then  $G$  is bipartite if and only if every edge has one endpoint in  $Ev(G, v)$  and the other in  $Odd(G, v)$ . Moreover, if  $G$  is bipartite, then the (unordered pair)  $\{Ev(G, v), Odd(G, v)\}$  is independent of  $v$ . That is,

$$\forall v, w \in V(G), \quad \{Ev(G, v), Odd(G, v)\} = \{Ev(G, w), Odd(G, w)\}.$$

Now suppose  $\tau = (B, h)$  is a type, and  $H \subseteq V(C(B, h))$ . Consider the question: Can  $H$  be the image of a bipartite graph?

Consider the condition

(15) For any  $i, j \in Ind(r)$ , either

1. for every  $x = (x_1, \dots, x_r) \in H$ ,  $x_i - x_j$  is even, or
2. for every  $x = (x_1, \dots, x_r) \in H$ ,  $x_i - x_j$  is odd.

As noted above, if  $H$  derives from a bipartite graph, then (15) is true.

Conversely, suppose  $H$  satisfies the subgraph coordinate condition and (15). Let  $x = (x_1, \dots, x_r)$  and  $y = (y_1, \dots, y_r)$  belong to  $H$  such that  $\mu(x, y) = 1$ . The latter means that there is at least one  $i \in Ind(r)$  for which  $|x_i - y_i| = 1$ . But (15) easily implies that  $|x_j - y_j| = 1$  for every  $j \in Ind(r)$ . It follows that *any* selection of edges which make  $H$  into a graph (on which the code map is the identity) produces a bipartite graph.

We can actually weaken condition (15). Fix an index  $i_0 \in Ind(r)$ . Suppose that for each  $x = (x_1, \dots, x_r) \in H$  and  $j \in Ind(r)$  for which  $x_j \neq 0$ , there is  $y = (y_1, \dots, y_r) \in H$  such that

$$(16.a) \quad \mu(x, y) = 1,$$

$$(16.b) \quad y_j = x_j - 1, \text{ and}$$

$$(16.c) \quad y_{i_0} \neq x_{i_0}.$$

Consider making  $H$  a graph using only edges generated by the above restrictions. On a casual reading, it appears that we must omit many of the edges allowed between

members of  $H$ . But this choice of edges makes  $H$  bipartite, partitioned by  $Ev(H, e_{i_0})$  and  $Odd(H, e_{i_0})$ .) Consequently,  $H$  satisfies (15).

The above comments suggest that the subgraph coordinate condition is subtler than its definition suggests.

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